4. $(X_1, \dots, X_n)$, $X_i \sim N(\mu_0, \sigma_0^2)$ and $(Y_1, \dots, Y_m)$, $Y_i \sim N(\mu_1, \sigma_1^2)$.

where all parameters are unknown. That is,

$$\Omega = \{ \theta = (\mu_0, \mu_1, \sigma_0, \sigma_1)', \ \mu_0, \mu_1 \in \mathbb{R} \text{ and } \sigma_0, \sigma_1 > 0 \}.$$

$H: \sigma_0^2 = \sigma_1^2$, i.e the variances of the samples in both pools are equal.

$A: \sigma_0^2 \neq \sigma_1^2$.

The hypothesis specifies the set $\omega = \{ \theta \in \Omega, \ \sigma_0 = \sigma_1 \}$.

As computed in problem 1, we have MLE's for $\mu_0, \mu_1, \sigma_0, \sigma_1$ under Ω.

$$\hat{\sigma}^2_{0,\Omega} = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2 \quad \text{and} \quad \hat{\sigma}^2_{1,\Omega} = \frac{1}{m} \sum_{i=1}^{m} (y_i - \bar{y})^2$$

since $\hat{\mu}_0 = \hat{\mu}_{0,\Omega} = \hat{\mu}_{0,\omega} = \bar{x}$ and $\hat{\mu}_1 = \hat{\mu}_{1,\Omega} = \hat{\mu}_{1,\omega} = \bar{y}$ since the restriction to ω does not affect μ_0 or μ_1.

We do, however have to compute the MLE for $\sigma_0 = \sigma_1 = \sigma$ under ω

The likelihood function is given by (letting $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_m)$)

$$L_\omega(\mu_0, \mu_1, \sigma \mid x, y) = \left(\frac{1}{\sqrt{2\pi}\, \sigma} \right)^{m+n} \exp\left(-\frac{1}{2\sigma^2} \left(\sum_{i=1}^{n} (x_i - \mu_0)^2 + \sum_{i=1}^{m} (y_i - \mu_1)^2 \right) \right)$$

So $$\log L = -(m+n) \log(\sqrt{2\pi}\, \sigma) - \frac{1}{2\sigma^2} \left(\sum_{i=1}^{n} (x_i - \mu_0)^2 + \sum_{i=1}^{m} (y_i - \mu_1)^2 \right)$$

and $$\frac{\partial}{\partial \sigma} \log L = \frac{-(m+n)}{\sigma} + \frac{1}{\sigma^3} \left(\sum_{i=1}^{n} (x_i - \mu_0)^2 + \sum_{i=1}^{m} (y_i - \mu_1)^2 \right)$$

$$= \frac{1}{\sigma} \left\{ \frac{1}{\sigma^2} \left(\sum_{i=1}^{n} (x_i - \mu_0)^2 + \sum_{i=1}^{m} (y_i - \mu_1)^2 \right) - (m+n) \right\}.$$

Thus we have a critical point when $\sigma^2 = \frac{1}{m+n} \left(\sum_{i=1}^{n} (x_i - \mu_0)^2 + \sum_{i=1}^{m} (y_i - \mu_1)^2 \right) := \hat{\sigma}^2_\omega$

is the MLE for σ once we verify that $\frac{\partial^2}{\partial \sigma^2} \log L < 0$ at $\sigma^2 = \hat{\sigma}^2_\omega$.

$$\frac{\partial^2}{\partial \sigma^2} \log L = \frac{(m+n)}{\sigma^2} - \frac{3}{\sigma^4} \left(\sum_{i=1}^{n} (x_i - \mu_0)^2 + \sum_{i=1}^{m} (y_i - \mu_1)^2 \right)$$

$$= \frac{1}{\sigma^2} \left((m+n) - \frac{3}{\sigma^2} \left(\sum_{i=1}^{n} (x_i - \mu_0)^2 + \sum_{i=1}^{m} (y_i - \mu_1)^2 \right) \right)$$ so evaluating at $\sigma^2 = \hat{\sigma}^2_\omega$,

we have $\left(\hat{\sigma}_w^2\right)^{-2}\Big((m+n) - 3(m+n)\Big) < 0$ since $\hat{\sigma}_w^2 > 0$ and $(m+n) - 3(m+n) < 0$. Thus $\hat{\sigma}_w^2$ is the MLE of $\sigma_0 = \sigma_1 = \sigma$ under ω.

Now we have likelihood functions $L(\hat{\Omega}) = L_\Omega(\hat{\mu}_0, \hat{\mu}_1, \hat{\sigma}_0^2, \hat{\sigma}_1^2 \mid x, y)$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^{m+n}\left(\frac{1}{\left(\hat{\sigma}^2_{0,\Omega}\right)^{n/2}\left(\hat{\sigma}^2_{1,\Omega}\right)^{m/2}}\right)\exp\left(-\frac{1}{2}\left(\frac{1}{\hat{\sigma}^2_{1,\Omega}}\sum_{i=1}^{n}(x_i-\hat{\mu}_0)^2 + \frac{1}{\hat{\sigma}^2_{2,\Omega}}\sum_{i=1}^{m}(y_i-\hat{\mu}_1)^2\right)\right)$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^{m+n}\left(\frac{1}{\left(\hat{\sigma}^2_{0,\Omega}\right)^{n/2}\left(\hat{\sigma}^2_{1,\Omega}\right)^{m/2}}\right)\exp\left(-\frac{1}{2}(n+m)\right).$$

and $L(\hat{\omega}) = L_\omega(\hat{\mu}_0, \hat{\mu}_1, \hat{\sigma}^2 \mid x, y) = \left(\frac{1}{\sqrt{2\pi}}\right)^{m+n}\left(\frac{1}{\left(\hat{\sigma}_w^2\right)^{(n+m)/2}}\right)\exp\left(\frac{-1}{2\hat{\sigma}_w^2}\left(\sum_{i=1}^{n}(x_i-\hat{\mu}_0)^2 + \sum_{i=1}^{m}(y_i-\hat{\mu}_1)^2\right)\right)$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^{m+n}\left(\frac{1}{\left(\hat{\sigma}_w^2\right)^{(n+m)/2}}\right)\exp\left(-\frac{1}{2}(n+m)\right).$$

Thus the likelihood ratio is $\dfrac{L(\hat{\omega})}{L(\hat{\Omega})} = \dfrac{\left(\hat{\sigma}^2_{0,\Omega}\right)^{n/2}\left(\hat{\sigma}^2_{1,\Omega}\right)^{m/2}}{\left(\hat{\sigma}_w^2\right)^{(m+n)/2}}$

$$= \frac{(n+m)^{(n+m)/2}\left(\dfrac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{\sum_{i=1}^{m}(y_i-\bar{y})^2}\right)^{n/2}}{n^{n/2}m^{m/2}\left(\dfrac{\sum_{i=1}^{n}(x_i-\bar{x})^2 + \sum_{i=1}^{m}(y_i-\bar{y})^2}{\sum_{i=1}^{m}(y_i-\bar{y})^2}\right)^{(n+m)/2}}$$

which we can manipulate to resemble the fisher distribution from problem 2, ...

$$= \sqrt{\frac{(n+m)^{n+m}}{n^n m^m}}\cdot\frac{\left(\dfrac{(n-1)\sum_{i=1}^{n}(x_i-\bar{x})^2/(n-1)}{(m-1)\sum_{i=1}^{m}(y_i-\bar{y})^2/(m-1)}\right)^{n/2}}{\left(1 + \dfrac{(n-1)\sum_{i=1}^{n}(x_i-\bar{x})^2/(n-1)}{(m-1)\sum_{i=1}^{m}(y_i-\bar{y})^2/(m-1)}\right)^{(n+m)/2}}.$$

$$= \sqrt{\frac{(n+m)^{n+m}}{n^n m^m}}\cdot\frac{\left(\left(\frac{n-1}{m-1}\right)f\right)^{n/2}}{\left(1+\left(\frac{n-1}{m-1}\right)f\right)} := A, \text{ if } f = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2/(n-1)}{\sum_{i=1}^{m}(y_i-\bar{y})^2/(m-1)}$$ and f is distributed

as $F_{n-1,m-1}$. Thus the test is given by $\phi(x,y) = \begin{cases} 1 & \text{if } A < C \\ 0 & \text{otherwise.} \end{cases}$

$$A = A(f) = \sqrt{\frac{(n+m)^{n+m}}{n^n m^m} \cdot \frac{\left(\left(\frac{n-1}{m-1}\right) f\right)^{n/2}}{\left(1 + \left(\frac{n-1}{m-1}\right) f\right)^{(n+m)/2}}}, \quad \text{so } A(0) = 0 \text{ and}$$

$\lim_{f \to \infty} A(f) = 0$ so A must attain a maximum for some $\tilde{f} \in [0, \infty)$.

$$\frac{\partial A}{\partial f} = \sqrt{\frac{(n+m)^{n+m}}{n^n m^m}} \left(\left(1 + \left(\frac{n-1}{m-1}\right) f\right)^{n+m}\right)^{-1} \left\{ \left(\frac{n}{2}\right)\left(\left(\frac{n-1}{m-1}\right) f\right)^{n/2 - 1} \left(1 + \left(\frac{n-1}{m-1}\right) f\right)^{n+m/2} \left(\frac{n-1}{m-1}\right) \right.$$

$$\left. - \left(\frac{n+m}{2}\right)\left(1 + \left(\frac{n-1}{m-1}\right) f\right)^{n+m/2 - 1} \left(\left(\frac{n-1}{m-1}\right) f\right)^{n/2} \left(\frac{n-1}{m-1}\right) \right\} = 0$$

$$\text{iff} \quad \left(\frac{n+m}{2}\right)\left(1 + \left(\frac{n-1}{m-1}\right) f\right)^{n+m/2 - 1} \left(\left(\frac{n-1}{m-1}\right) f\right)^{n/2} = \left(\frac{n}{2}\right)\left(\left(\frac{n-1}{m-1}\right) f\right)^{n/2 - 1} \left(1 + \left(\frac{n-1}{m-1}\right) f\right)^{n+m/2}$$

$$\Rightarrow \left(\frac{n+m}{2}\right)\left(\frac{n-1}{m-1}\right) f = \left(\frac{n}{2}\right)\left(1 + \left(\frac{n-1}{m-1}\right) f\right) \Rightarrow \frac{n}{2}\left(\frac{n-1}{m-1}\right) f + \frac{m}{2}\left(\frac{n-1}{m-1}\right) f = \frac{n}{2} + \frac{n}{2}\left(\frac{n-1}{m-1}\right) f$$

$\Rightarrow \frac{m}{2}\left(\frac{n-1}{m-1}\right) f = \frac{n}{2} \Rightarrow \tilde{f} = \frac{n}{m}\left(\frac{m-1}{n-1}\right)$ is a maximum since it's the only critical point in $[0, \infty)$ and A is strictly positive.

Thus for constants c_1 and c_2, $A(f) < C$ iff $f < c_1$ or $f > c_2$ since A in general attains C for two values of f. That is, $A(c_1) = A(c_2)$; $c_1 < c_2$.

Thus the test is given by $\phi(x,y) = \begin{cases} 1 & \text{if } f < c_1 \text{ or } f > c_2 \\ 0 & \text{otherwise} \end{cases}$

where c_1 and c_2 are uniquely determined by

$$P(F_{n-1,m-1} < c_1) = P(F_{n-1,m-1} > c_2) = \alpha/2.$$